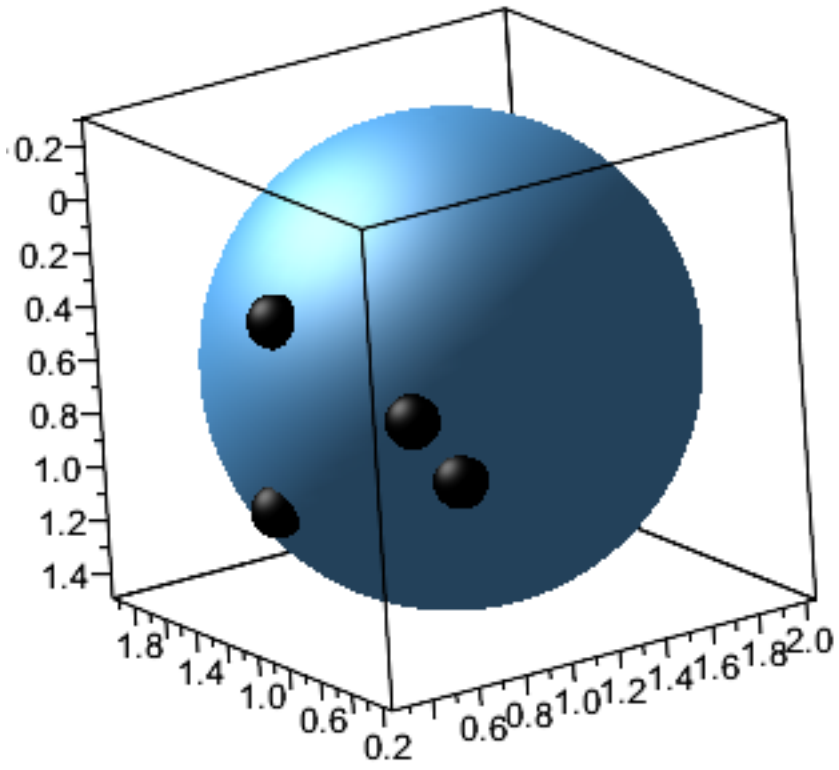
Center and Radius of Sphere Given Four Points

Introduction

This application will derive symbolic expressions that gives the center point (x_c, y_c, z_c) and radius r of a sphere whose surface passes through four known (but symbolic) points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4)



The solve command is used to rearrange some seemingly simple equations. The resulting equations, however, are large, and would be difficult to derive by hand. This demonstrates Maple's utility in deriving and manipulating large symbolic expressions.

Equations

Each of these four equations describes the relationship between the center (x_c, y_c, z_c) and radius (r) of a circle, and a known point (x_n, y_n)

> $eq_1 := (x_1 - x_c)^2 + (y_1 - y_c)^2 + (z_1 - z_c)^2 = r^2$:
$eq_2 := (x_2 - x_c)^2 + (y_2 - y_c)^2 + (z_2 - z_c)^2 = r^2$:
$eq_3 := (x_3 - x_c)^2 + (y_3 - y_c)^2 + (z_3 - z_c)^2 = r^2$:

$$eq_4 := (x_4 - x_c)^2 + (y_4 - y_c)^2 + (z_4 - z_c)^2 = r^2 :$$

Symbolic Solution

> $res := solve([eq_1, eq_2, eq_3, eq_4], [x_c, y_c, z_c, r])[\,]$:

Hence the coordinates of the center are

> $res_1;$

$res_2;$

$res_3;$

$$x_c = \frac{1}{2}\left(x_1^2 y_2 z_3 - x_1^2 y_2 z_4 - x_1^2 y_3 z_2 + x_1^2 y_3 z_4 + x_1^2 y_4 z_2 - x_1^2 y_4 z_3 - x_2^2 y_1 z_3 + x_2^2 y_1 z_4 + x_2^2 y_3 z_1 - x_2^2 y_3 z_4 - x_2^2 y_4 z_1 + x_2^2 y_4 z_3 + x_3^2 y_1 z_2 - x_3^2 y_1 z_4 - x_3^2 y_2 z_1 + x_3^2 y_2 z_4 + x_3^2 y_4 z_1 - x_3^2 y_4 z_2 - x_4^2 y_1 z_2 + x_4^2 y_1 z_3 + x_4^2 y_2 z_1 - x_4^2 y_2 z_3 - x_4^2 y_3 z_1 + x_4^2 y_3 z_2 + y_1^2 y_2 z_3 - y_1^2 y_2 z_4 - y_1^2 y_3 z_2 + y_1^2 y_3 z_4 + y_1^2 y_4 z_2 - y_1^2 y_4 z_3 - y_1 y_2^2 z_3 + y_1 y_2^2 z_4 + y_1 y_3^2 z_2 - y_1 y_3^2 z_4 - y_1 y_4^2 z_2 + y_1 y_4^2 z_3 - y_1 z_2^2 z_3 + y_1 z_2^2 z_4 + y_1 z_2 z_3^2 - y_1 z_2 z_4^2 - y_1 z_3^2 z_4 + y_1 z_3 z_4^2 + y_2^2 y_3 z_1 - y_2^2 y_3 z_4 - y_2^2 y_4 z_1 + y_2^2 y_4 z_3 - y_2 y_3^2 z_1 + y_2 y_3^2 z_4 + y_2 y_4^2 z_1 - y_2 y_4^2 z_3 + y_2 z_1^2 z_3 - y_2 z_1^2 z_4 - y_2 z_1 z_3^2 + y_2 z_1 z_4^2 + y_2 z_3^2 z_4 - y_2 z_3 z_4^2 + y_3^2 y_4 z_1 - y_3^2 y_4 z_2 - y_3 y_4^2 z_1 + y_3 y_4^2 z_2 - y_3 z_1^2 z_2 + y_3 z_1^2 z_4 + y_3 z_1 z_2^2 - y_3 z_1 z_4^2 - y_3 z_2^2 z_4 + y_3 z_2 z_4^2 + y_4 z_1^2 z_2 - y_4 z_1^2 z_3 - y_4 z_1 z_2^2 + y_4 z_1 z_3^2 + y_4 z_2^2 z_3 - y_4 z_2 z_3^2\right) \Big/ \left(x_1 y_2 z_3 - x_1 y_2 z_4 - x_1 y_3 z_2 + x_1 y_3 z_4 + x_1 y_4 z_2 - x_1 y_4 z_3 - x_2 y_1 z_3 + x_2 y_1 z_4 + x_2 y_3 z_1 - x_2 y_3 z_4 - x_2 y_4 z_1 + x_2 y_4 z_3 + x_3 y_1 z_2 - x_3 y_1 z_4 - x_3 y_2 z_1 + x_3 y_2 z_4 + x_3 y_4 z_1 - x_3 y_4 z_2 - x_4 y_1 z_2 + x_4 y_1 z_3 + x_4 y_2 z_1 - x_4 y_2 z_3 - x_4 y_3 z_1 + x_4 y_3 z_2\right)$$

$$y_c = -\frac{1}{2}\left(x_1^2 x_2 z_3 - x_1^2 x_2 z_4 - x_1^2 x_3 z_2 + x_1^2 x_3 z_4 + x_1^2 x_4 z_2 - x_1^2 x_4 z_3 - x_1 x_2^2 z_3 + x_1 x_2^2 z_4 + x_1 x_3^2 z_2 - x_1 x_3^2 z_4 - x_1 x_4^2 z_2 + x_1 x_4^2 z_3 - x_1 y_2^2 z_3 + x_1 y_2^2 z_4 + x_1 y_3^2 z_2 - x_1 y_3^2 z_4 - x_1 y_4^2 z_2 + x_1 y_4^2 z_3 - x_1 z_2^2 z_3 + x_1 z_2^2 z_4 + x_1 z_2 z_3^2 - x_1 z_2 z_4^2 - x_1 z_3^2 z_4 + x_1 z_3 z_4^2 + x_2^2 x_3 z_1 - x_2^2 x_3 z_4 - x_2^2 x_4 z_1 + x_2^2 x_4 z_3 - x_2 x_3^2 z_1 + x_2 x_3^2 z_4 + x_2 x_4^2 z_1 - x_2 x_4^2 z_3 + x_2 y_1^2 z_3 - x_2 y_1^2 z_4 - x_2 y_3^2 z_1 + x_2 y_3^2 z_4 + x_2 y_4^2 z_1 - x_2 y_4^2 z_3 + x_2 z_1^2 z_3 - x_2 z_1^2 z_4 - x_2 z_1 z_3^2 + x_2 z_1 z_4^2 + x_2 z_3^2 z_4 - x_2 z_3 z_4^2 + x_3^2 x_4 z_1 - x_3^2 x_4 z_2 - x_3 x_4^2 z_1 + x_3 x_4^2 z_2 - x_3 y_1^2 z_2 + x_3 y_1^2 z_4 + x_3 y_2^2 z_1 - x_3 y_2^2 z_4 - x_3 y_4^2 z_1 + x_3 y_4^2 z_2 - x_3 z_1^2 z_2 + x_3 z_1^2 z_4 + x_3 z_1 z_2^2 - x_3 z_1 z_4^2 - x_3 z_2^2 z_4 + x_3 z_2 z_4^2 + x_4 y_1^2 z_2 - x_4 y_1^2 z_3 - x_4 y_2^2 z_1 + x_4 y_2^2 z_3 + x_4 y_3^2 z_1 - x_4 y_3^2 z_2 + x_4 z_1^2 z_2 - x_4 z_1^2 z_3 - x_4 z_1 z_2^2 + x_4 z_1 z_3^2 + x_4 z_2^2 z_3 - x_4 z_2 z_3^2\right) \Big/ \left(x_1 y_2 z_3 - x_1 y_2 z_4 - x_1 y_3 z_2 + x_1 y_3 z_4 + x_1 y_4 z_2 - x_1 y_4 z_3 - x_2 y_1 z_3 + x_2 y_1 z_4 + x_2 y_3 z_1 - x_2 y_3 z_4 - x_2 y_4 z_1 + x_2 y_4 z_3 + x_3 y_1 z_2 - x_3 y_1 z_4 - x_3 y_2 z_1 + x_3 y_2 z_4 + x_3 y_4 z_1 - x_3 y_4 z_2 - x_4 y_1 z_2 + x_4 y_1 z_3 + x_4 y_2 z_1 - x_4 y_2 z_3 - x_4 y_3 z_1 + x_4 y_3 z_2\right)$$

$$z_c = \frac{1}{2}\left(x_1^2 x_2 y_3 - x_1^2 x_2 y_4 - x_1^2 x_3 y_2 + x_1^2 x_3 y_4 + x_1^2 x_4 y_2 - x_1^2 x_4 y_3 - x_1 x_2^2 y_3 + x_1 x_2^2 y_4 + x_1 x_3^2 y_2 - x_1 x_3^2 y_4 - x_1 x_4^2 y_2 + x_1 x_4^2 y_3 - x_1 y_2^2 y_3 + x_1 y_2^2 y_4 + x_1 y_2 y_3^2 - x_1 y_2 y_4^2 + x_1 y_2 z_3^2 - x_1 y_2 z_4^2 \right. \tag{3.1}$$

$$-x_1y_3^2y_4+x_1y_3y_4^2-x_1y_3z_2^2+x_1y_3z_4^2+x_1y_4z_2^2-x_1y_4z_3^2+x_2^2x_3y_1-x_2^2x_3y_4-x_2^2x_4y_1$$
$$+x_2^2x_4y_3-x_2x_3^2y_1+x_2x_3^2y_4+x_2x_4^2y_1-x_2x_4^2y_3+x_2y_1^2y_3-x_2y_1^2y_4-x_2y_1y_3^2+x_2y_1y_4^2$$
$$-x_2y_1z_3^2+x_2y_1z_4^2+x_2y_3^2y_4-x_2y_3y_4^2+x_2y_3z_1^2-x_2y_3z_4^2-x_2y_4z_1^2+x_2y_4z_3^2+x_3^2x_4y_1$$
$$-x_3^2x_4y_2-x_3x_4^2y_1+x_3x_4^2y_2-x_3y_1^2y_2+x_3y_1^2y_4+x_3y_1y_2^2-x_3y_1y_4^2+x_3y_1z_2^2-x_3y_1z_4^2$$
$$-x_3y_2^2y_4+x_3y_2y_4^2-x_3y_2z_1^2+x_3y_2z_4^2+x_3y_4z_1^2-x_3y_4z_2^2+x_4y_1^2y_2-x_4y_1^2y_3-x_4y_1y_2^2$$
$$+x_4y_1y_3^2-x_4y_1z_2^2+x_4y_1z_3^2+x_4y_2^2y_3-x_4y_2y_3^2+x_4y_2z_1^2-x_4y_2z_3^2-x_4y_3z_1^2+x_4y_3z_2^2\Big)\Big/$$
$$\left(x_1y_2z_3-x_1y_2z_4-x_1y_3z_2+x_1y_3z_4+x_1y_4z_2-x_1y_4z_3-x_2y_1z_3+x_2y_1z_4+x_2y_3z_1-x_2y_3z_4\right.$$
$$-x_2y_4z_1+x_2y_4z_3+x_3y_1z_2-x_3y_1z_4-x_3y_2z_1+x_3y_2z_4+x_3y_4z_1-x_3y_4z_2-x_4y_1z_2+x_4y_1z_3$$
$$\left.+x_4y_2z_1-x_4y_2z_3-x_4y_3z_1+x_4y_3z_2\right)$$

The radius of the sphere is

> $R := simplify\left(allvalues\left(res_4\right)_2\right)$

$R := r$ **(3.2)**

$$= \Big(\left(\left(y_3^2-2y_3y_4+y_4^2+\left(z_3-z_4\right)^2\right)x_2^2+\left(\left(\left(-2y_3+2y_4\right)y_2+2y_3y_4-2y_4^2-2\left(z_3\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-z_4\right)\left(z_2-z_4\right)\right)x_3+2x_4\left(\left(y_3-y_4\right)y_2-y_3^2+y_3y_4+\left(z_3-z_4\right)\left(-z_3+z_2\right)\right)\right)x_2+\left(y_2^2\right.$$
$$-2y_2y_4+y_4^2+\left(z_2-z_4\right)^2\Big)x_3^2-2x_4\left(y_2^2+\left(-y_3-y_4\right)y_2+y_3y_4+\left(z_2-z_4\right)\left(-z_3+z_2\right)\right)x_3$$
$$+\left(y_2^2-2y_2y_3+y_3^2+\left(-z_3+z_2\right)^2\right)x_4^2+\left(\left(-z_3+z_4\right)y_2+\left(z_2-z_4\right)y_3-y_4\left(-z_3+z_2\right)\right)^2\Big)x_1^4$$
$$+\Big(\left(-2y_3^2+4y_3y_4-2y_4^2-2\left(z_3-z_4\right)^2\right)x_2^3+\left(\left(\left(2y_3-2y_4\right)y_2-2y_3y_4+2y_4^2+2\left(z_3\right.\right.\right.$$
$$\left.\left.\left.-z_4\right)\left(z_2-z_4\right)\right)x_3-2x_4\left(\left(y_3-y_4\right)y_2-y_3^2+y_3y_4+\left(z_3-z_4\right)\left(-z_3+z_2\right)\right)\right)x_2^2+\left(\left(\left(2y_3\right.\right.\right.$$
$$\left.\left.-2y_4\right)y_2-2y_3y_4+2y_4^2+2\left(z_3-z_4\right)\left(z_2-z_4\right)\right)x_3^2+\left(\left(-2y_3+2y_4\right)y_2+2y_3^2-2y_3y_4\right.$$
$$\left.-2\left(z_3-z_4\right)\left(-z_3+z_2\right)\right)x_4^2-2\left(y_2^2+\left(-y_3-y_4\right)y_2+y_3y_4+\left(z_2-z_4\right)\left(-z_3+z_2\right)\right)\left(y_3^2\right.$$
$$\left.-2y_3y_4+y_4^2+\left(z_3-z_4\right)^2\right)\Big)x_2+\left(-2y_2^2+4y_2y_4-2y_4^2-2\left(z_2-z_4\right)^2\right)x_3^3+2x_4\left(y_2^2+\left(\right.\right.$$
$$\left.\left.-y_3-y_4\right)y_2+y_3y_4+\left(z_2-z_4\right)\left(-z_3+z_2\right)\right)x_3^2+\left(\left(2y_2^2+\left(-2y_3-2y_4\right)y_2+2y_3y_4+2\left(z_2\right.\right.\right.$$
$$\left.\left.\left.-z_4\right)\left(-z_3+z_2\right)\right)x_4^2+2\left(y_2^2-2y_2y_4+y_4^2+\left(z_2-z_4\right)^2\right)\left(\left(y_3-y_4\right)y_2-y_3^2+y_3y_4+\left(z_3\right.\right.\right.$$
$$\left.\left.\left.-z_4\right)\left(-z_3+z_2\right)\right)\right)x_3-2\left(x_4^2+\left(y_3-y_4\right)y_2-y_3y_4+y_4^2+\left(z_3-z_4\right)\left(z_2-z_4\right)\right)x_4\left(y_2^2\right.$$
$$\left.-2y_2y_3+y_3^2+\left(-z_3+z_2\right)^2\right)\Big)x_1^3+\Big(\left(y_3^2-2y_3y_4+y_4^2+\left(z_3-z_4\right)^2\right)x_2^4+\left(\left(\left(2y_3\right.\right.\right.$$
$$\left.\left.\left.-2y_4\right)y_1-2y_3y_4+2y_4^2-2\left(z_4-z_1\right)\left(z_3-z_4\right)\right)x_3+2x_4\left(\left(-y_3+y_4\right)y_1+y_3^2-y_3y_4+\left(z_3\right.\right.\right.$$

$$-z_1)\,(z_3-z_4)\,)\,)\,x_2^{\,3}+\Big(\,(\,(\,-2y_2-2y_3+4y_4)\,y_1+(\,-2y_3+4y_4)\,y_2+4y_3y_4-6y_4^{\,2}+(\,-2z_2-2z_3+4z_4)\,z_1+(\,-2z_3+4z_4)\,z_2+4z_3z_4-6z_4^{\,2}\Big)\,x_3^{\,2}-2\,(\,(\,-2y_2+y_3+y_4)\,y_1+(y_3+y_4)\,y_2-2y_3y_4+(\,-2z_2+z_3+z_4)\,z_1+(z_3+z_4)\,z_2-2z_3z_4)\,x_4x_3+\Big(\,(\,-2y_2+4y_3-2y_4)\,y_1+(4y_3-2y_4)\,y_2-6y_3^{\,2}+4y_3y_4+(\,-2z_2+4z_3-2z_4)\,z_1+(4z_3-2z_4)\,z_2-6z_3^{\,2}+4z_3z_4\Big)\,x_4^{\,2}+\Big(2y_3^{\,2}-4y_3y_4+2y_4^{\,2}+2\,(z_3-z_4)^{\,2}\Big)\,y_1^{\,2}+\Big(-2\,(z_3-z_4)^{\,2}y_2-2y_3^{\,3}+2y_3^{\,2}y_4+\Big(2y_4^{\,2}+2\,(z_3-z_4)\,(\,-z_3+z_2)\Big)\,y_3-2y_4\Big(y_4^{\,2}+(z_3-z_4)\,(z_2-z_4)\Big)\Big)\,y_1+\Big(2y_3^{\,2}-4y_3y_4+2y_4^{\,2}+2\,(z_3-z_4)^{\,2}\Big)\,y_2^{\,2}+\Big(-2y_3^{\,3}+2y_3^{\,2}y_4+\Big(2y_4^{\,2}-2\,(z_3-z_1)\,(z_3-z_4)\Big)\,y_3+2\,\Big(-y_4^{\,2}+(z_4-z_1)\,(z_3-z_4)\Big)\,y_4\Big)\,y_2+4y_3^{\,3}y_4+\Big(-8y_4^{\,2}+2z_1^{\,2}+(\,-2z_2-2z_3)\,z_1+2z_2^{\,2}-2z_2z_3+4z_3z_4-2z_4^{\,2}\Big)\,y_3^{\,2}-4\left(-y_4^{\,2}+z_1^{\,2}+\left(-z_2-\frac{1}{2}\,z_3-\frac{1}{2}\,z_4\right)z_1+z_2^{\,2}+\left(-\frac{1}{2}\,z_3-\frac{1}{2}\,z_4\right)z_2-z_3^{\,2}+3z_3z_4-z_4^{\,2}\right)\,y_4y_3+\Big(2z_1^{\,2}+(\,-2z_2-2z_4)\,z_1+2z_2^{\,2}-2z_2z_4-2z_3\,(z_3-2z_4)\Big)\,y_4^{\,2}+2\,\Big(z_1^{\,2}+(\,-z_3-z_4)\,z_1+z_2^{\,2}+(\,-z_3-z_4)\,z_2+2z_3z_4\Big)\,(z_3-z_4)^{\,2}\Big)\,x_2^{\,2}+\Big(\,(\,(2y_2-2y_4)\,y_1-2y_2y_4+2y_4^{\,2}-2\,(z_4-z_1)\,(z_2-z_4)\Big)\,x_3^{\,3}-2x_4\,(\,(y_2-2y_3+y_4)\,y_1+(y_3-2y_4)\,y_2+y_3y_4+(z_2-2z_3+z_4)\,z_1+(z_3-2z_4)\,z_2+z_3z_4)\,x_3^{\,2}+\Big(\,(\,(\,-2y_2-2y_3+4y_4)\,y_1+(4y_3-2y_4)\,y_2-2y_3y_4+(\,-2z_2-2z_3+4z_4)\,z_1+(4z_3-2z_4)\,z_2-2z_3z_4)\,x_4^{\,2}+\Big(\,(\,-4y_3+4y_4)\,y_2+4y_3y_4-4y_4^{\,2}-4\,(z_3-z_4)\,(z_2-z_4)\Big)\,y_1^{\,2}+\Big(\,(2y_3-2y_4)\,y_2^{\,2}+\Big(2y_3^{\,2}+2\,(z_3-z_4+y_4)\,(z_3-z_4-y_4)\Big)\,y_2-2y_3^{\,2}y_4+2\,(z_2-z_4+y_4)\,(z_2-z_4-y_4)\,y_3-2\,\Big(-2y_4^{\,2}+z_2^{\,2}+(\,-4z_3+2z_4)\,z_2+z_3^{\,2}+2z_3z_4-2z_4^{\,2}\Big)\,y_4\Big)\,y_1+\Big(-2y_3y_4+2y_4^{\,2}-2\,(z_4-z_1)\,(z_3-z_4)\Big)\,y_2^{\,2}+\Big(-2y_3^{\,2}y_4-4\,(z_4+y_4-z_1)\,(z_4-y_4-z_1)\,y_3-2y_4\Big(y_4^{\,2}-2z_1^{\,2}+(2z_3+2z_4)\,z_1+z_3^{\,2}-4z_3z_4+z_4^{\,2}\Big)\Big)\,y_2+\Big(2y_4^{\,2}-2\,(z_4-z_1)\,(z_2-z_4)\Big)\,y_3^{\,2}-2\,\Big(y_4^{\,2}-2z_1^{\,2}+(2z_2+2z_4)\,z_1+z_2^{\,2}-4z_2z_4+z_4^{\,2}\Big)\,y_4y_3+\Big(-4z_1^{\,2}+(2z_2+2z_3+4z_4)\,z_1+2z_2^{\,2}+(\,-4z_3-2z_4)\,z_2+2z_3\,(z_3-z_4)\Big)\,y_4^{\,2}-2\,(z_4-z_1)\,(z_3-z_4)\,(z_2-z_4)\,(z_2+z_3-2z_1)\,)\,x_3+2x_4\Big(\,(\,(y_2-y_3)\,y_1-y_2y_3+y_3^{\,2}-(z_3-z_1)\,(\,-z_3+z_2)\Big)\,x_4^{\,2}+\Big(\,(2y_3-2y_4)\,y_2-2y_3^{\,2}+2y_3y_4+2\,(z_3-z_4)\,(\,-z_3+z_2)\Big)\,y_1^{\,2}+\Big(\,(\,-y_3$$

$$+y_4)\,y_2^{\,2}+\left(-y_3^{\,2}+y_4^{\,2}+\left(z_3-z_4\right)^2\right)y_2+2y_3^{\,3}-y_3^{\,2}y_4+\left(-y_4^{\,2}-z_2^{\,2}+\left(-2z_3+4z_4\right)z_2\right.$$

$$\left.+2z_3^{\,2}-2z_3z_4-z_4^{\,2}\right)y_3+y_4\left(-z_3+z_2\right)^2\Big)y_1+\left(y_3^{\,2}-y_3y_4+\left(z_3-z_1\right)\left(z_3-z_4\right)\right)y_2^{\,2}+\left(-y_3^{\,3}\right.$$

$$\left.+2y_3^{\,2}y_4+\left(-y_4^{\,2}+2z_1^{\,2}+\left(-2z_3-2z_4\right)z_1-z_3^{\,2}+4z_3z_4-z_4^{\,2}\right)y_3-2y_4\left(z_3-z_1\right)^2\right)y_2$$

$$-y_3^{\,3}y_4+\left(y_4^{\,2}-2z_1^{\,2}+\left(z_2+2z_3+z_4\right)z_1+z_2^{\,2}+\left(-z_3-2z_4\right)z_2-z_4\left(z_3-z_4\right)\right)y_3^{\,2}-\left(-2z_1^{\,2}\right.$$

$$\left.+\left(2z_2+2z_3\right)z_1+z_2^{\,2}-4z_2z_3+z_3^{\,2}\right)y_4y_3+\left(-z_3+z_2\right)\left(z_3-z_1\right)\left(-y_4^{\,2}+\left(z_3-z_4\right)\left(z_2+z_4\right.\right.$$

$$\left.\left.-2z_1\right)\right)\Big)\Big)x_2+\left(y_2^{\,2}-2y_2y_4+y_4^{\,2}+\left(z_2-z_4\right)^2\right)x_3^{\,4}+2\left(\left(-y_2+y_4\right)y_1+y_2^{\,2}-y_2y_4+\left(z_2\right.\right.$$

$$\left.\left.-z_1\right)\left(z_2-z_4\right)\right)x_4x_3^{\,3}+\Big(\left(\left(4y_2-2y_3-2y_4\right)y_1-6y_2^{\,2}+\left(4y_3+4y_4\right)y_2-2y_3y_4+\left(4z_2\right.\right.$$

$$\left.\left.-2z_3-2z_4\right)z_1-6z_2^{\,2}+\left(4z_3+4z_4\right)z_2-2z_3z_4\right)x_4^{\,2}+\left(2y_2^{\,2}-4y_2y_4+2y_4^{\,2}+2\left(z_2\right.\right.$$

$$\left.\left.-z_4\right)^2\right)y_1^{\,2}+\left(-2y_2^{\,3}+2y_2^{\,2}y_4+\left(2y_4^{\,2}-2\left(z_2-z_4\right)\left(-z_3+z_2\right)\right)y_2-2\left(z_2-z_4\right)^2y_3\right.$$

$$\left.-2y_4\left(y_4^{\,2}+\left(z_3-z_4\right)\left(z_2-z_4\right)\right)\right)y_1+\left(-2y_3+4y_4\right)y_2^{\,3}+\left(2y_3^{\,2}+2y_3y_4-8y_4^{\,2}+2z_1^{\,2}+\left(\right.\right.$$

$$\left.\left.-2z_2-2z_3\right)z_1+\left(-2z_3+4z_4\right)z_2+2z_3^{\,2}-2z_4^{\,2}\right)y_2^{\,2}+\Big(-4y_3^{\,2}y_4+\left(2y_4^{\,2}-2\left(z_2-z_1\right)\left(z_2\right.\right.$$

$$\left.\left.-z_4\right)\right)y_3+4\left(y_4^{\,2}-z_1^{\,2}+\left(\frac{1}{2}z_2+z_3+\frac{1}{2}z_4\right)z_1+z_2^{\,2}+\left(\frac{1}{2}z_3-3z_4\right)z_2-z_3^{\,2}+\frac{1}{2}z_3z_4\right.$$

$$\left.\left.+z_4^{\,2}\right)y_4\right)y_2+\left(2y_4^{\,2}+2\left(z_2-z_4\right)^2\right)y_3^{\,2}+2\left(-y_4^{\,2}+\left(z_4-z_1\right)\left(z_2-z_4\right)\right)y_4y_3+\left(2z_1^{\,2}+\left(\right.\right.$$

$$\left.\left.-2z_3-2z_4\right)z_1-2z_2^{\,2}+4z_2z_4+2z_3\left(z_3-z_4\right)\right)y_4^{\,2}-2\left(z_2-z_4\right)^2\left(-z_1^{\,2}+\left(z_2+z_4\right)z_1+\left(z_3\right.\right.$$

$$\left.\left.-2z_4\right)z_2-z_3\left(z_3-z_4\right)\right)\Big)x_3^{\,2}-2\Big(\left(\left(y_2-y_3\right)y_1-y_2^{\,2}+y_2y_3-\left(z_2-z_1\right)\left(-z_3+z_2\right)\right)x_4^{\,2}$$

$$+\left(2y_2^{\,2}+\left(-2y_3-2y_4\right)y_2+2y_3y_4+2\left(z_2-z_4\right)\left(-z_3+z_2\right)\right)y_1^{\,2}+\left(-2y_2^{\,3}+\left(y_3+y_4\right)y_2^{\,2}\right.$$

$$\left.+\left(y_3^{\,2}+y_4^{\,2}-2z_2^{\,2}+\left(2z_3+2z_4\right)z_2+z_3^{\,2}-4z_3z_4+z_4^{\,2}\right)y_2-y_3^{\,2}y_4+\left(-y_4^{\,2}-\left(z_2-z_4\right)^2\right)y_3\right.$$

$$\left.-y_4\left(-z_3+z_2\right)^2\right)y_1+\left(y_3+y_4\right)y_2^{\,3}+\left(-y_3^{\,2}-2y_3y_4-y_4^{\,2}+2z_1^{\,2}+\left(-2z_2-z_3-z_4\right)z_1+\left(z_3\right.\right.$$

$$\left.\left.+z_4\right)z_2-\left(z_3-z_4\right)^2\right)y_2^{\,2}+\left(y_3^{\,2}y_4+\left(y_4^{\,2}-2z_1^{\,2}+\left(2z_2+2z_4\right)z_1+z_2^{\,2}-4z_2z_4+z_4^{\,2}\right)y_3+\left(\right.\right.$$

$$\left.\left.-2z_1^{\,2}+\left(2z_2+2z_3\right)z_1+z_2^{\,2}-4z_2z_3+z_3^{\,2}\right)y_4\right)y_2+\Big(\left(-z_2+z_4\right)y_3^{\,2}+2y_4\left(z_2-z_1\right)y_3+\left(-z_3\right.$$

$$\left.\left.+z_2\right)\left(-y_4^{\,2}+\left(z_3+z_4-2z_1\right)\left(z_2-z_4\right)\right)\right)\left(z_2-z_1\right)\Big)x_4x_3+\left(y_2^{\,2}-2y_2y_3+y_3^{\,2}+\left(-z_3\right.\right.$$

$$\left.\left.+z_2\right)^2\right)x_4^{\,4}+\Big(\left(2y_2^{\,2}-4y_2y_3+2y_3^{\,2}+2\left(-z_3+z_2\right)^2\right)y_1^{\,2}+\left(-2y_2^{\,3}+2y_2^{\,2}y_3+\left(2y_3^{\,2}-2\left(z_2\right.\right.\right.$$

$$-z_4)\left(-z_3+z_2\right)\Big)y_2-2y_3^{\,3}+2\left(z_3-z_4\right)\left(-z_3+z_2\right)y_3-2y_4\left(-z_3+z_2\right)^2\Big)y_1+\Big(4y_3$$

$$-2y_4\Big)y_2^{\,3}+\Big(-8y_3^{\,2}+2y_3y_4+2y_4^{\,2}+2z_1^{\,2}+\left(-2z_2-2z_4\right)z_1+\left(4z_3-2z_4\right)z_2-2z_3^{\,2}$$

$$+2z_4^{\,2}\Big)y_2^{\,2}+\Big(4y_3^{\,3}+2y_3^{\,2}y_4+\Big(-4y_4^{\,2}-4z_1^{\,2}+\left(2z_2+2z_3+4z_4\right)z_1+4z_2^{\,2}+\Big(-12z_3$$

$$+2z_4\Big)z_2+4z_3^{\,2}+2z_3z_4-4z_4^{\,2}\Big)y_3-2y_4\left(z_2-z_1\right)\left(-z_3+z_2\right)\Big)y_2-2y_3^{\,3}y_4+\Big(2y_4^{\,2}+2z_1^{\,2}$$

$$+\left(-2z_3-2z_4\right)z_1-2z_2^{\,2}+4z_2z_3-2z_4\left(z_3-z_4\right)\Big)y_3^{\,2}+2y_4\left(z_3-z_1\right)\left(-z_3+z_2\right)y_3$$

$$+4\left(\frac{1}{2}y_4^{\,2}+\frac{1}{2}z_1^{\,2}+\left(-\frac{1}{2}z_2-\frac{1}{2}z_3\right)z_1+\left(z_3-\frac{1}{2}z_4\right)z_2-\frac{1}{2}z_4\left(z_3-z_4\right)\right)\left(-z_3+z_2\right)^2\Big)x_4^{\,2}$$

$$+2\left(\left(-z_3+z_4\right)y_2+\left(z_2-z_4\right)y_3-y_4\left(-z_3+z_2\right)\right)^2y_1^{\,2}+2\Big(\left(-z_3+z_4\right)y_2+\left(z_2-z_4\right)y_3-y_4\Big($$

$$-z_3+z_2\Big)\Big)\Big(\left(z_3-z_4\right)y_2^{\,2}+\left(-z_2+z_4\right)y_3^{\,2}+\left(-z_3+z_2\right)\left(y_4^{\,2}+\left(z_3-z_4\right)\left(z_2-z_4\right)\right)\Big)y_1+\Big(y_3^{\,2}$$

$$-2y_3y_4+y_4^{\,2}+\left(z_3-z_4\right)^2\Big)y_2^{\,4}+\Big(-2y_3^{\,3}+2y_3^{\,2}y_4+\Big(2y_4^{\,2}-2\left(z_3-z_1\right)\left(z_3-z_4\right)\Big)y_3+2\Big($$

$$-y_4^{\,2}+\left(z_4-z_1\right)\left(z_3-z_4\right)\Big)y_4\Big)y_2^{\,3}+\Big(y_3^{\,4}+2y_3^{\,3}y_4+\Big(-6y_4^{\,2}+\left(-2z_2-2z_3+4z_4\right)z_1+2z_2^{\,2}$$

$$-2z_2z_3+2z_3^{\,2}-2z_4^{\,2}\Big)y_3^{\,2}-4\left(-\frac{1}{2}y_4^{\,2}+\left(-z_2+\frac{1}{2}z_3+\frac{1}{2}z_4\right)z_1+z_2^{\,2}+\left(-\frac{1}{2}z_3-\frac{1}{2}z_4\right)z_2$$

$$-\frac{1}{2}\left(z_3-z_4\right)^2\Big)y_4y_3+y_4^{\,4}+\Big(\left(-2z_2+4z_3-2z_4\right)z_1+2z_2^{\,2}-2z_2z_4-2z_3^{\,2}+2z_4^{\,2}\Big)y_4^{\,2}$$

$$+2\left(z_1^{\,2}+\left(-z_3-z_4\right)z_1+z_2^{\,2}+\left(-z_3-z_4\right)z_2+\frac{1}{2}\left(z_3+z_4\right)^2\right)\left(z_3-z_4\right)^2\Big)y_2^{\,2}+\Big(-2y_4y_3^{\,4}$$

$$+\left(2y_4^{\,2}-2\left(z_2-z_1\right)\left(z_2-z_4\right)\right)y_3^{\,3}+2y_4\Big(y_4^{\,2}+\left(-z_2+2z_3-z_4\right)z_1+z_2^{\,2}+\left(z_3-2z_4\right)z_2$$

$$-2\left(z_3+\frac{1}{2}z_4\right)\left(z_3-z_4\right)\Big)y_3^{\,2}+\Big(-2y_4^{\,4}+\Big(\left(-2z_2-2z_3+4z_4\right)z_1+2z_2^{\,2}+\left(-4z_3+2z_4\right)z_2$$

$$+2\left(z_3+2z_4\right)\left(z_3-z_4\right)\Big)y_4^{\,2}-2\left(2z_1^{\,2}+\left(-z_2-z_3-2z_4\right)z_1+z_2z_3+z_4^{\,2}\right)\left(z_2-z_4\right)\left(z_3\right.$$

$$\left.-z_4\right)\Big)y_3+2\left(-z_3+z_2\right)y_4\Big(\left(-z_2+z_1\right)y_4^{\,2}+\Big(2z_1^{\,2}+\left(-z_2-2z_3-z_4\right)z_1+z_2z_4+z_3^{\,2}\Big)\left(z_3\right.$$

$$\left.-z_4\right)\Big)\Big)y_2+\Big(y_4^{\,2}+\left(z_2-z_4\right)^2\Big)y_3^{\,4}+2\left(-y_4^{\,2}+\left(z_4-z_1\right)\left(z_2-z_4\right)\right)y_4y_3^{\,3}+\Big(y_4^{\,4}+\Big(\left(4z_2\right.$$

$$\left.-2z_3-2z_4\right)z_1-2z_2^{\,2}+2z_3^{\,2}-2z_3z_4+2z_4^{\,2}\Big)y_4^{\,2}+\left(z_2-z_4\right)^2\Big(2z_1^{\,2}+\left(-2z_2-2z_4\right)z_1+z_2^{\,2}$$

$$+\left(-2z_3+2z_4\right)z_2+2z_3^{\,2}-2z_3z_4+z_4^{\,2}\Big)\Big)y_3^{\,2}-2\left(-z_3+z_2\right)y_4\Big(\left(-z_3+z_1\right)y_4^{\,2}+\left(z_2\right.$$

$$\left.-z_4\right)\Big(2z_1^{\,2}+\left(-2z_2-z_3-z_4\right)z_1+z_2^{\,2}+z_3z_4\Big)\Big)y_3+\left(-z_3+z_2\right)^2\Big(y_4^{\,4}+\Big(2z_1^{\,2}+\left(-2z_2\right.$$

$$\left.-2z_3\right)z_1+z_2^{\,2}+\left(2z_3-2z_4\right)z_2+z_3^{\,2}-2z_3z_4+2z_4^{\,2}\Big)y_4^{\,2}+\left(z_3-z_4\right)^2\left(z_2-z_4\right)^2\Big)\Big)x_1^{\,2}$$

$$+\Big(\Big(\Big(\big(-2y_3+2y_4\big)y_1+2y_3y_4-2y_4^2+2\big(z_4-z_1\big)\big(z_3-z_4\big)\Big)x_3-2x_4\Big(\big(-y_3+y_4\big)y_1+y_3^2$$

$$-y_3y_4+\big(z_3-z_1\big)\big(z_3-z_4\big)\Big)\Big)x_2^4+\Big(\Big(\big(2y_3-2y_4\big)y_1-2y_3y_4+2y_4^2-2\big(z_4-z_1\big)\big(z_3$$

$$-z_4\big)\Big)x_3^2+\Big(\big(-2y_3+2y_4\big)y_1+2y_3^2-2y_3y_4+2\big(z_3-z_1\big)\big(z_3-z_4\big)\Big)x_4^2-2\Big(y_1^2+\big(-y_3$$

$$-y_4\big)y_1+y_3y_4+\big(z_4-z_1\big)\big(z_3-z_1\big)\Big)\Big(y_3^2-2y_3y_4+y_4^2+\big(z_3-z_4\big)^2\Big)\Big)x_2^3+\Big(\Big(\big(2y_2$$

$$-2y_4\big)y_1-2y_2y_4+2y_4^2-2\big(z_4-z_1\big)\big(z_2-z_4\big)\Big)x_3^3+4\Big(\Big(-\frac{1}{2}y_2-\frac{1}{2}y_3+y_4\Big)y_1+\Big(y_3$$

$$-\frac{1}{2}y_4\Big)y_2-\frac{1}{2}y_3y_4+\Big(-\frac{1}{2}z_2-\frac{1}{2}z_3+z_4\Big)z_1+\Big(z_3-\frac{1}{2}z_4\Big)z_2-\frac{1}{2}z_3z_4\Big)x_4x_3^2+\Big(\Big(\big(-2y_2$$

$$+4y_3-2y_4\big)y_1+\big(-2y_3+4y_4\big)y_2-2y_3y_4+\big(-2z_2+4z_3-2z_4\big)z_1+\big(-2z_3+4z_4\big)z_2$$

$$-2z_3z_4\big)x_4^2+\Big(\big(2y_3-2y_4\big)y_2-2y_3y_4+2y_4^2+2\big(z_3-z_4\big)\big(z_2-z_4\big)\Big)y_1^2+\Big(\big(-4y_3$$

$$+4y_4\big)y_2^2+\big(2y_3^2+2\big(z_3-z_4+y_4\big)\big(z_3-z_4-y_4\big)\big)y_2-2y_3^2y_4-4\big(z_2-z_4+y_4\big)\big(z_2-z_4$$

$$-y_4\big)y_3+4\Big(-\frac{1}{2}y_4^2+z_2^2+\big(-z_3-z_4\big)z_2-\frac{1}{2}z_3^2+2z_3z_4-\frac{1}{2}z_4^2\Big)y_4\Big)y_1+\big(4y_3y_4-4y_4^2$$

$$+4\big(z_4-z_1\big)\big(z_3-z_4\big)\big)y_2^2+\big(-2y_3^2y_4+2\big(z_4+y_4-z_1\big)\big(z_4-y_4-z_1\big)y_3-2y_4\big(-2y_4^2$$

$$+z_1^2+\big(-4z_3+2z_4\big)z_1+z_3^2+2z_3z_4-2z_4^2\big)\big)y_2+\big(2y_4^2-2\big(z_4-z_1\big)\big(z_2-z_4\big)\big)y_3^2+4\Big($$

$$-\frac{1}{2}y_4^2-\frac{1}{2}z_1^2+\big(-z_2+2z_4\big)z_1+z_2^2-z_2z_4-\frac{1}{2}z_4^2\Big)y_4y_3+\big(2z_1^2+\big(2z_2-4z_3-2z_4\big)z_1$$

$$-4z_2^2+\big(2z_3+4z_4\big)z_2+2z_3\big(z_3-z_4\big)\big)y_4^2+4\big(z_4-z_1\big)\Big(z_2-\frac{1}{2}z_3-\frac{1}{2}z_1\Big)\big(z_2-z_4\big)\big(z_3$$

$$-z_4\big)\Big)x_3-4\Big(\Big(\Big(\Big(\frac{1}{2}y_3-\frac{1}{2}y_2\Big)y_1+\frac{1}{2}y_2y_3-\frac{1}{2}y_3^2+\frac{1}{2}\big(z_3-z_1\big)\big(-z_3+z_2\big)\Big)x_4^2+\Big(\Big(\frac{1}{2}y_3$$

$$-\frac{1}{2}y_4\Big)y_2-\frac{1}{2}y_3^2+\frac{1}{2}y_3y_4+\frac{1}{2}\big(z_3-z_4\big)\big(-z_3+z_2\big)\Big)y_1^2+\Big(\big(-y_3+y_4\big)y_2^2+\Big(\frac{1}{2}y_3^2$$

$$-\frac{1}{2}y_4^2-\frac{1}{2}\big(z_3-z_4\big)^2\Big)y_2+\frac{1}{2}y_3^3-y_3^2y_4+\Big(\frac{1}{2}y_4^2-z_2^2+\big(z_3+z_4\big)z_2-2z_3z_4+\frac{1}{2}z_4^2$$

$$+\frac{1}{2}z_3^2\Big)y_3+y_4\big(-z_3+z_2\big)^2\Big)y_1+\big(y_3^2-y_3y_4+\big(z_3-z_1\big)\big(z_3-z_4\big)\big)y_2^2+\Big(-y_3^3+\frac{1}{2}y_3^2y_4$$

$$+\Big(\frac{1}{2}y_4^2+\frac{1}{2}z_1^2+\big(z_3-2z_4\big)z_1-z_3^2+z_3z_4+\frac{1}{2}z_4^2\Big)y_3-\frac{1}{2}y_4\big(z_3-z_1\big)^2\Big)y_2+\frac{1}{2}y_3^3y_4+\Big($$

$$-\frac{1}{2}y_4^2-\frac{1}{2}z_1^2+\Big(-\frac{1}{2}z_2+\frac{1}{2}z_3+z_4\Big)z_1+z_2^2+\Big(-z_3-\frac{1}{2}z_4\Big)z_2+\frac{1}{2}z_4\big(z_3-z_4\big)\Big)y_3^2-\Big($$

$$-\frac{1}{2}z_1^2+\big(-z_2+2z_3\big)z_1+z_2^2-z_2z_3-\frac{1}{2}z_3^2\Big)y_4y_3+\big(-z_3+z_2\big)\big(z_3-z_1\big)\Big(\frac{1}{2}y_4^2+\Big(z_2-\frac{1}{2}z_4$$

$$-\frac{1}{2}z_1\Big)\big(z_3-z_4\big)\Big)\Big)x_4\Big)x_2^2+\Big(\Big(\big(-2y_2+2y_4\big)y_1+2y_2y_4-2y_4^2+2\big(z_4-z_1\big)\big(z_2-z_4\big)\Big)x_3^4$$

$$+\Big(\big((4y_2-2y_3-2y_4)y_1+(-2y_3-2y_4)y_2+4y_3y_4+(4z_2-2z_3-2z_4)z_1+(-2z_3-2z_4)z_2+4z_3z_4\big)x_4^2+\big((2y_3-2y_4)y_2-2y_3y_4+2y_4^2+2(z_3-z_4)(z_2-z_4)\big)y_1^2+\big((2y_3-2y_4)y_2^2+\big(-4y_3^2-4(z_3-z_4+y_4)(z_3-z_4-y_4)\big)y_2+4y_3^2y_4+2(z_2-z_4+y_4)(z_2-z_4-y_4)y_3-2y_4\big(y_4^2+z_2^2+(2z_3-4z_4)z_2-2z_3^2+2z_3z_4+z_4^2\big)\big)y_1+\big(-2y_3y_4+2y_4^2-2(z_4-z_1)(z_3-z_4)\big)y_2^2+\Big(4y_3^2y_4+2(z_4+y_4-z_1)(z_4-y_4-z_1)y_3+4y_4\Big(-\frac{1}{2}y_4^2-\frac{1}{2}z_1^2+(-z_3+2z_4)z_1+z_3^2-z_3z_4-\frac{1}{2}z_4^2\Big)\Big)y_2+\big(-4y_4^2+4(z_4-z_1)(z_2-z_4)\big)y_3^2-2\big(-2y_4^2+z_1^2+(-4z_2+2z_4)z_1+z_2^2+2z_2z_4-2z_4^2\big)y_4y_3+\big(2z_1^2+(-4z_2+2z_3-2z_4)z_1+2z_2^2+(2z_3-2z_4)z_2-4z_3(z_3-z_4)\big)y_4^2-2(z_4-z_1)(z_3-z_4)(z_2-z_4)(z_2-2z_3+z_1)\big)x_3^2+\big(\big(-2y_2+2y_3\big)y_1+2y_2y_3-2y_3^2+2(z_3-z_1)(-z_3+z_2)\big)x_4^4+\big(\big((-2y_3+2y_4)y_2+2y_3^2-2y_3y_4-2(z_3-z_4)(-z_3+z_2)\big)y_1^2+\big((-2y_3+2y_4)y_2^2+\big(4y_3^2-4y_4^2-4(z_3-z_4)^2\big)y_2-2y_3^3-2y_3^2y_4+\big(4y_4^2-2z_2^2+(8z_3-4z_4)z_2-2z_3^2-4z_3z_4+4z_4^2\big)y_3+2y_4(-z_3+z_2)^2\big)y_1+\big(2y_3^2-2y_3y_4+2(z_3-z_1)(z_3-z_4)\big)y_2^2+\big(-2y_3^3-2y_3^2y_4+\big(4y_4^2-2z_1^2+(8z_3-4z_4)z_1-2z_3^2-4z_3z_4+4z_4^2\big)y_3+2y_4(z_3-z_1)^2\big)y_2+4y_3^3y_4+\big(-4y_4^2+2z_1^2+(-4z_2-2z_3+2z_4)z_1+2z_2^2+(-2z_3+2z_4)z_2+4z_4(z_3-z_4)\big)y_3^2-2\big(z_1^2+(-4z_2+2z_3)z_1+z_2^2+2z_2z_3-2z_3^2\big)y_4y_3+2\big(2y_4^2+(z_3-z_4)(z_2-2z_4+z_1)\big)(-z_3+z_2)(z_3-z_1)\big)x_4^2-2\big(\big(y_2^2+(-y_3-y_4)y_2+y_3y_4+(z_2-z_4)(-z_3+z_2)\big)y_1^2+\big((-y_3-y_4)y_2^2+\big(y_3^2+2y_3y_4+y_4^2+(z_3-z_4)^2\big)y_2-y_3^2y_4+\big(-y_4^2-(z_2-z_4)^2\big)y_3-y_4(-z_3+z_2)^2\big)y_1+\big(y_3y_4+(z_4-z_1)(z_3-z_1)\big)y_2^2+\big(-y_3^2y_4+\big(-y_4^2-(z_4-z_1)^2\big)y_3-y_4(z_3-z_1)^2\big)y_2+\big(y_4^2-(z_4-z_1)(z_2-z_4)\big)y_3^2+y_4(z_2-z_1)^2y_3+(-z_3+z_2)(z_3-z_1)\big(-y_4^2+(z_4-z_1)(z_2-z_4)\big)\big)\big(y_3^2-2y_3y_4+y_4^2+(z_3-z_4)^2\big)\big)x_2-2\big(\big(-y_2+y_4\big)y_1+y_2^2-y_2y_4+(z_2-z_1)(z_2-z_4)\big)x_4x_3^4+\big(\big((-2y_2+2y_4)y_1+2y_2^2-2y_2y_4+2(z_2-z_1)(z_2-z_4)\big)x_4^2-2\big(y_2^2-2y_2y_4+y_4^2+(z_2-z_4)^2\big)\big(y_1^2+(-y_2-y_4)y_1+y_2y_4+(z_4-z_1)(z_2-z_1)\big)\big)x_3^3$$

$$+4\left(\left(\left(\left(\frac{1}{2}y_3-\frac{1}{2}y_2\right)y_1+\frac{1}{2}y_2^2-\frac{1}{2}y_2y_3+\frac{1}{2}\left(z_2-z_1\right)\left(-z_3+z_2\right)\right)x_4^2+\left(\frac{1}{2}y_2^2+\left(-\frac{1}{2}y_3\right.\right.\right.\right.$$
$$\left.\left.-\frac{1}{2}y_4\right)y_2+\frac{1}{2}y_3y_4+\frac{1}{2}\left(z_2-z_4\right)\left(-z_3+z_2\right)\right)y_1^2+\left(-\frac{1}{2}y_2^3+\left(-\frac{1}{2}y_3+y_4\right)y_2^2+\left(y_3^2\right.\right.$$
$$\left.-\frac{1}{2}y_4^2-\frac{1}{2}z_2^2+\left(-z_3+2z_4\right)z_2+z_3^2-z_3z_4-\frac{1}{2}z_4^2\right)y_2-y_3^2y_4+\left(\frac{1}{2}y_4^2+\frac{1}{2}\left(z_2\right.\right.$$
$$\left.\left.-z_4\right)^2\right)y_3-y_4\left(-z_3+z_2\right)^2\Big)y_1+\left(y_3-\frac{1}{2}y_4\right)y_2^3+\left(-y_3^2-\frac{1}{2}y_3y_4+\frac{1}{2}y_4^2+\frac{1}{2}z_1^2+\left(\right.\right.$$
$$\left.-\frac{1}{2}z_2+\frac{1}{2}z_3-z_4\right)z_1+\left(z_3-\frac{1}{2}z_4\right)z_2-\left(z_3+\frac{1}{2}z_4\right)\left(z_3-z_4\right)\Big)y_2^2+\left(y_3^2y_4+\left(-\frac{1}{2}y_4^2\right.\right.$$
$$\left.\left.-\frac{1}{2}z_1^2+\left(-z_2+2z_4\right)z_1+z_2^2-z_2z_4-\frac{1}{2}z_4^2\right)y_3-\frac{1}{2}\left(z_1^2+\left(-4z_2+2z_3\right)z_1+z_2^2+2z_2z_3\right.\right.$$
$$\left.\left.-2z_3^2\right)y_4\right)y_2+\Big(\left(-z_2+z_4\right)y_3^2+\frac{1}{2}y_4\left(z_2-z_1\right)y_3+\left(-z_3+z_2\right)\left(\frac{1}{2}y_4^2+\left(z_3-\frac{1}{2}z_4\right.\right.\right.$$
$$\left.\left.\left.-\frac{1}{2}z_1\right)\left(z_2-z_4\right)\right)\right)\left(z_2-z_1\right)\Big)x_4x_3^2+\Big(\left(\left(2y_2-2y_3\right)y_1-2y_2^2+2y_2y_3-2\left(z_2-z_1\right)\left(-z_3\right.\right.$$
$$\left.\left.+z_2\right)\right)x_4^4+\Big(\left(2y_2^2+\left(-2y_3-2y_4\right)y_2+2y_3y_4+2\left(z_2-z_4\right)\left(-z_3+z_2\right)\right)y_1^2+\left(-2y_2^3\right.$$
$$+\left(4y_3-2y_4\right)y_2^2+\left(-2y_3^2+4y_4^2-2z_2^2+\left(8z_3-4z_4\right)z_2-2z_3^2-4z_3z_4+4z_4^2\right)y_2$$
$$+2y_3^2y_4+\left(-4y_4^2-4\left(z_2-z_4\right)^2\right)y_3+2y_4\left(-z_3+z_2\right)^2\Big)y_1+\left(-2y_3+4y_4\right)y_2^3+\left(2y_3^2\right.$$
$$\left.-2y_3y_4-4y_4^2+2z_1^2+\left(-2z_2-4z_3+2z_4\right)z_1+\left(-2z_3+4z_4\right)z_2+2\left(z_3+2z_4\right)\left(z_3-z_4\right)\right)$$
$$y_2^2+\Big(-2y_3^2y_4+\left(4y_4^2-2z_1^2+\left(8z_2-4z_4\right)z_1-2z_2^2-4z_2z_4+4z_4^2\right)y_3+4\Big(-\frac{1}{2}z_1^2+\left(\right.$$
$$\left.-z_2+2z_3\right)z_1+z_2^2-z_2z_3-\frac{1}{2}z_3^2\Big)y_4\Big)y_2-2\Big(\left(-z_2+z_4\right)y_3^2-y_4\left(z_2-z_1\right)y_3+\left(-z_3\right.$$
$$\left.\left.+z_2\right)\left(2y_4^2+\left(z_3-2z_4+z_1\right)\left(z_2-z_4\right)\right)\right)\left(z_2-z_1\right)\Big)x_4^2+2\Big(\left(\left(y_3-y_4\right)y_2-y_3^2+y_3y_4\right.$$
$$\left.+\left(z_3-z_4\right)\left(-z_3+z_2\right)\right)y_1^2+\Big(\left(-y_3+y_4\right)y_2^2+\left(y_3^2-2y_3y_4+y_4^2+\left(z_3-z_4\right)^2\right)y_2+y_3^2y_4$$
$$+\left(-y_4^2-\left(z_2-z_4\right)^2\right)y_3+y_4\left(-z_3+z_2\right)^2\Big)y_1+\left(y_3y_4-y_4^2+\left(z_4-z_1\right)\left(z_3-z_4\right)\right)y_2^2+\Big(\right.$$
$$-y_3^2y_4+\left(y_4^2+\left(z_4-z_1\right)^2\right)y_3-y_4\left(z_3-z_1\right)^2\Big)y_2+\Big(\left(-z_4+z_1\right)y_3^2+y_4\left(z_2-z_1\right)y_3+\left(-y_4^2\right.$$
$$\left.\left.+\left(z_4-z_1\right)\left(z_3-z_4\right)\right)\left(-z_3+z_2\right)\right)\left(z_2-z_1\right)\Big)\left(y_2^2-2y_2y_4+y_4^2+\left(z_2-z_4\right)^2\right)\Big)x_3-2\Big(\left(y_1^2\right.$$
$$+\left(-y_2-y_3\right)y_1+y_2y_3+\left(z_3-z_1\right)\left(z_2-z_1\right)\Big)x_4^2+\Big(\left(y_3-y_4\right)y_2-y_3y_4+y_4^2+\left(z_3-z_4\right)\left(z_2\right.$$
$$\left.-z_4\right)\Big)y_1^2+\Big(\left(-y_3+y_4\right)y_2^2+\left(-y_3^2+2y_3y_4-y_4^2-\left(z_3-z_4\right)^2\right)y_2+y_3^2y_4+\left(-y_4^2\right.$$
$$\left.-\left(z_2-z_4\right)^2\right)y_3+y_4\left(-z_3+z_2\right)^2\Big)y_1+\left(y_3^2-y_3y_4+\left(z_3-z_1\right)\left(z_3-z_4\right)\right)y_2^2+\Big(-y_3^2y_4$$

$$+\left(y_4^2+\left(z_4-z_1\right)^2\right)y_3-y_4\left(z_3-z_1\right)^2\Big)y_2+\Big(\left(z_2-z_4\right)y_3^2-y_4\left(z_2-z_1\right)y_3+\left(z_3-z_1\right)\left(y_4^2\right.$$

$$\left.+\left(z_3-z_4\right)\left(z_2-z_4\right)\right)\Big)\left(z_2-z_1\right)\Big)x_4\left(y_2^2-2y_2y_3+y_3^2+\left(-z_3+z_2\right)^2\right)\Big)x_1+\Big(\Big(y_1^2-2y_1y_4$$

$$+y_4^2+\left(z_4-z_1\right)^2\Big)x_3^2-2x_4\left(y_1^2+\left(-y_3-y_4\right)y_1+y_3y_4+\left(z_4-z_1\right)\left(z_3-z_1\right)\right)x_3+\left(y_1^2\right.$$

$$-2y_1y_3+y_3^2+\left(z_3-z_1\right)^2\Big)x_4^2+\Big(\left(z_3-z_4\right)y_1+\left(z_4-z_1\right)y_3-y_4\left(z_3-z_1\right)\Big)^2\Big)x_2^4+\Big(\Big(-2y_1^2$$

$$+4y_1y_4-2y_4^2-2\left(z_4-z_1\right)^2\Big)x_3^3+2x_4\left(y_1^2+\left(-y_3-y_4\right)y_1+y_3y_4+\left(z_4-z_1\right)\left(z_3\right.\right.$$

$$\left.-z_1\right)\Big)x_3^2+\Big(\Big(2y_1^2+\left(-2y_3-2y_4\right)y_1+2y_3y_4+2\left(z_4-z_1\right)\left(z_3-z_1\right)\Big)x_4^2-2\left(y_1^2-2y_1y_4\right.$$

$$\left.+y_4^2+\left(z_4-z_1\right)^2\right)\Big(\left(-y_3+y_4\right)y_1+y_3^2-y_3y_4+\left(z_3-z_1\right)\left(z_3-z_4\right)\Big)\Big)x_3+2x_4\left(y_1^2-2y_1y_3\right.$$

$$\left.+y_3^2+\left(z_3-z_1\right)^2\right)\Big(-x_4^2+\left(-y_3+y_4\right)y_1+y_3y_4-y_4^2+\left(z_4-z_1\right)\left(z_3-z_4\right)\Big)\Big)x_2^3+\Big(\Big(y_1^2$$

$$-2y_1y_4+y_4^2+\left(z_4-z_1\right)^2\Big)x_3^4+2x_4\left(y_1^2+\left(-y_2-y_4\right)y_1+y_2y_4+\left(z_4-z_1\right)\left(z_2-z_1\right)\Big)x_3^3$$

$$+\Big(\Big(-6y_1^2+\left(4y_2+4y_3+4y_4\right)y_1+\left(-2y_3-2y_4\right)y_2-2y_3y_4-6z_1^2+\left(4z_2+4z_3\right.$$

$$\left.+4z_4\right)z_1+\left(-2z_3-2z_4\right)z_2-2z_3z_4\Big)x_4^2+\left(-2y_2-2y_3+4y_4\right)y_1^3+\left(2y_2^2+2y_2y_4+2y_3^2\right.$$

$$\left.+2y_3y_4-8y_4^2+\left(-2z_2-2z_3+4z_4\right)z_1+2z_2^2-2z_2z_3+2z_3^2-2z_4^2\right)y_1^2+\Big(-4y_2^2y_4$$

$$+\left(2y_4^2-2\left(z_4-z_1\right)\left(z_3-z_1\right)\right)y_2-4y_3^2y_4+\left(2y_4^2-2\left(z_4-z_1\right)\left(z_2-z_1\right)\right)y_3-4\Big(-y_4^2$$

$$-z_1^2+\left(-\frac{1}{2}z_2-\frac{1}{2}z_3+3z_4\right)z_1+z_2^2+\left(-z_3-\frac{1}{2}z_4\right)z_2+z_3^2-\frac{1}{2}z_3z_4-z_4^2\Big)y_4\Big)y_1+\left(2y_4^2\right.$$

$$\left.+2\left(z_4-z_1\right)^2\right)y_2^2+\Big(-2\left(z_4-z_1\right)^2y_3+2\left(-y_4^2+\left(z_4-z_1\right)\left(z_3-z_4\right)\right)y_4\Big)y_2+\left(2y_4^2\right.$$

$$\left.+2\left(z_4-z_1\right)^2\right)y_3^2+2\left(-y_4^2+\left(z_4-z_1\right)\left(z_2-z_4\right)\right)y_4y_3+\Big(-2z_1^2+4z_4z_1+2z_2^2+\left(-2z_3\right.$$

$$\left.-2z_4\right)z_2+2z_3\left(z_3-z_4\right)\Big)y_4^2+2\left(z_4-z_1\right)^2\Big(\left(-z_2-z_3+2z_4\right)z_1+z_2^2-z_2z_4+z_3\left(z_3-z_4\right)\Big)\Big)$$

$$x_3^2-4\Big(\Big(-\frac{1}{2}y_1^2+\left(\frac{1}{2}y_3+\frac{1}{2}y_2\right)y_1-\frac{1}{2}y_2y_3-\frac{1}{2}\left(z_3-z_1\right)\left(z_2-z_1\right)\Big)x_4^2+\Big(\frac{1}{2}y_3+\frac{1}{2}y_4$$

$$-y_2\Big)y_1^3+\Big(y_2^2+\left(\frac{1}{2}y_3+\frac{1}{2}y_4\right)y_2-\frac{1}{2}y_3^2-y_3y_4-\frac{1}{2}y_4^2+\left(-z_2+\frac{1}{2}z_3+\frac{1}{2}z_4\right)z_1+z_2^2$$

$$+\left(-\frac{1}{2}z_3-\frac{1}{2}z_4\right)z_2-\frac{1}{2}\left(z_3-z_4\right)^2\Big)y_1^2+\Big(\left(-y_3-y_4\right)y_2^2+\Big(\frac{1}{2}y_3^2+\frac{1}{2}y_4^2-z_1^2+\left(z_3\right.$$

$$\left.+z_4\right)z_1-2z_3z_4+\frac{1}{2}z_4^2+\frac{1}{2}z_3^2\Big)y_2+\frac{1}{2}y_3^2y_4+\Big(\frac{1}{2}y_4^2+\frac{1}{2}z_1^2+\left(z_2-2z_4\right)z_1+\frac{1}{2}z_4^2-z_2^2$$

$$+z_2z_4\Big)y_3-\Big(-\frac{1}{2}z_1^2+\left(-z_2+2z_3\right)z_1+z_2^2-z_2z_3-\frac{1}{2}z_3^2\Big)y_4\Big)y_1+\left(y_3y_4+\left(z_4-z_1\right)\left(z_3\right.\right.$$

$$-z_1)\,)\,y_2^{\,2}+\left(-\frac{1}{2}\,y_3^{\,2}y_4+\left(-\frac{1}{2}\,y_4^{\,2}-\frac{1}{2}\,(z_4-z_1)^{\,2}\right)y_3-\frac{1}{2}\,y_4\,(z_3-z_1)^{\,2}\right)y_2+\left(\left(-\frac{1}{2}\,z_4\right.\right.$$
$$\left.+\frac{1}{2}\,z_1\right)y_3^{\,2}+y_4\,(z_2-z_1)\,y_3+\left(-\frac{1}{2}\,y_4^{\,2}+(z_4-z_1)\left(z_2-\frac{1}{2}\,z_3-\frac{1}{2}\,z_4\right)\right)(z_3-z_1)\Big)\,(z_2$$
$$\left.-z_1)\right)x_4x_3+\left(y_1^{\,2}-2\,y_1y_3+y_3^{\,2}+(z_3-z_1)^{\,2}\right)x_4^{\,4}+\Big(\left(-2\,y_2+4\,y_3-2\,y_4\right)y_1^{\,3}+\left(2\,y_2^{\,2}\right.$$
$$+2\,y_2y_3-8\,y_3^{\,2}+2\,y_3y_4+2\,y_4^{\,2}+\left(-2\,z_2+4\,z_3-2\,z_4\right)z_1+2\,z_2^{\,2}-2\,z_2z_4-2\,z_3^{\,2}+2\,z_4^{\,2}\Big)\,y_1^{\,2}$$
$$+\Big(-4\,y_2^{\,2}y_3+\left(2\,y_3^{\,2}-2\,(z_4-z_1)\,(z_3-z_1)\right)y_2+4\,y_3^{\,3}+2\,y_3^{\,2}y_4+\Big(-4\,y_4^{\,2}+4\,z_1^{\,2}+\left(2\,z_2\right.$$
$$\left.-12\,z_3+2\,z_4\right)z_1-4\,z_2^{\,2}+\left(2\,z_3+4\,z_4\right)z_2+4\,z_3^{\,2}+2\,z_3z_4-4\,z_4^{\,2}\Big)\,y_3-2\,y_4\,(z_3-z_1)\,(z_2-z_1)\Big)$$
$$y_1+\left(2\,y_3^{\,2}+2\,(z_3-z_1)^{\,2}\right)y_2^{\,2}+\Big(-2\,y_3^{\,3}-2\,(z_3-z_1)\,(z_3-z_4)\,y_3-2\,y_4\,(z_3-z_1)^{\,2}\Big)\,y_2$$
$$-2\,y_3^{\,3}y_4+\Big(2\,y_4^{\,2}-2\,z_1^{\,2}+4\,z_3z_1+2\,z_2^{\,2}+\left(-2\,z_3-2\,z_4\right)z_2-2\,z_4\,(z_3-z_4)\Big)\,y_3^{\,2}+2\,y_4\,(z_3$$
$$-z_1)\,(-z_3+z_2)\,y_3+2\,(z_3-z_1)^{\,2}\left(y_4^{\,2}+\left(-z_2+2\,z_3-z_4\right)z_1+z_2^{\,2}-z_2z_3-z_4\,(z_3-z_4)\right)\Big)\,x_4^{\,2}$$
$$+\left(y_3^{\,2}-2\,y_3y_4+y_4^{\,2}+(z_3-z_4)^{\,2}\right)y_1^{\,4}+\Big(-2\,(z_3-z_4)^{\,2}y_2-2\,y_3^{\,3}+2\,y_3^{\,2}y_4+\Big(2\,y_4^{\,2}+2\,(z_3$$
$$-z_4)\,(-z_3+z_2)\Big)\,y_3-2\,y_4\left(y_4^{\,2}+(z_3-z_4)\,(z_2-z_4)\right)\Big)\,y_1^{\,3}+\Big(2\,(z_3-z_4)^{\,2}y_2^{\,2}+2\,\Big(\left(-z_4\right.$$
$$\left.+z_1)\,y_3+y_4\,(z_3-z_1)\right)\,(z_3-z_4)\,y_2+y_3^{\,4}+2\,y_3^{\,3}y_4+\Big(-6\,y_4^{\,2}+2\,z_1^{\,2}+\left(-2\,z_2-2\,z_3\right)z_1+\Big($$
$$-2\,z_3+4\,z_4\Big)\,z_2+2\,z_3^{\,2}-2\,z_4^{\,2}\Big)\,y_3^{\,2}-2\,\Big(-y_4^{\,2}+2\,z_1^{\,2}+\left(-2\,z_2-z_3-z_4\right)z_1+\left(z_3+z_4\right)z_2$$
$$-\,(z_3-z_4)^{\,2}\Big)\,y_4y_3+y_4^{\,4}+\Big(2\,z_1^{\,2}+\left(-2\,z_2-2\,z_4\right)z_1+\left(4\,z_3-2\,z_4\right)z_2-2\,z_3^{\,2}+2\,z_4^{\,2}\Big)\,y_4^{\,2}$$
$$+2\left(z_1^{\,2}+\left(-z_3-z_4\right)z_1+z_2^{\,2}+\left(-z_3-z_4\right)z_2+\frac{1}{2}\,(z_3+z_4)^{\,2}\right)(z_3-z_4)^{\,2}\Big)\,y_1^{\,2}+\Big(-4\,\Big(\left(-z_4\right.$$
$$\left.+z_1)\,y_3+y_4\,(z_3-z_1)\right)\,(z_3-z_4)\,y_2^{\,2}-2\,\Big((z_4-z_1)\,y_3^{\,2}+\left(-y_4^{\,2}+(z_4-z_1)\,(z_3-z_4)\right)(z_3$$
$$-z_1)\Big)\,(z_3-z_4)\,y_2-2\,y_4y_3^{\,4}+\left(2\,y_4^{\,2}-2\,(z_4-z_1)\,(z_2-z_1)\right)y_3^{\,3}+4\,y_4\Big(\frac{1}{2}\,y_4^{\,2}+\frac{1}{2}\,z_1^{\,2}+\Big($$
$$-\frac{1}{2}\,z_2+\frac{1}{2}\,z_3-z_4\Big)\,z_1+\left(z_3-\frac{1}{2}\,z_4\right)z_2-\left(z_3+\frac{1}{2}\,z_4\right)(z_3-z_4)\Big)\,y_3^{\,2}+\Big(-2\,y_4^{\,4}+\Big(2\,z_1^{\,2}+\Big($$
$$-2\,z_2-4\,z_3+2\,z_4\Big)\,z_1+\left(-2\,z_3+4\,z_4\right)z_2+2\,\left(z_3+2\,z_4\right)(z_3-z_4)\Big)\,y_4^{\,2}+4\,(z_4-z_1)\,\Big(\Big(\frac{1}{2}\,z_3$$
$$-\frac{1}{2}\,z_2\Big)\,z_1+z_2^{\,2}+\left(-\frac{1}{2}\,z_3-z_4\right)z_2+\frac{1}{2}\,z_4^{\,2}\Big)\,(z_3-z_4)\Big)\,y_3-4\,\Big(\Big(\frac{1}{2}\,z_2-\frac{1}{2}\,z_1\Big)\,y_4^{\,2}+\Big(\Big(-\frac{1}{2}\,z_2$$
$$+\frac{1}{2}\,z_4\Big)\,z_1+z_2^{\,2}+\left(-z_3-\frac{1}{2}\,z_4\right)z_2+\frac{1}{2}\,z_3^{\,2}\Big)\,(z_3-z_4)\Big)\,(z_3-z_1)\,y_4\Big)\,y_1+2\,\Big(\left(-z_4\right.$$
$$\left.+z_1)\,y_3+y_4\,(z_3-z_1)\right)^{\,2}y_2^{\,2}+2\,\Big((z_4-z_1)\,y_3^{\,2}+\left(-y_4^{\,2}+(z_4-z_1)\,(z_3-z_4)\right)(z_3-z_1)\Big)\,(($$

$$-z_4+z_1)\,y_3+y_4\,(z_3-z_1)\,)\,y_2+\left(y_4^{\,2}+(z_4-z_1)^{\,2}\right)y_3^{\,4}+2\left(-y_4^{\,2}+(z_4-z_1)\,(z_2-z_4)\right)y_4y_3^{\,3}$$
$$+\left(y_4^{\,4}+\left(-2\,z_1^{\,2}+4\,z_2z_1+(-2\,z_3-2\,z_4)\,z_2+2\,z_3^{\,2}-2\,z_3z_4+2\,z_4^{\,2}\right)y_4^{\,2}+2\,(z_4-z_1)^{\,2}\left(\frac{1}{2}\,z_1^{\,2}\right.\right.$$
$$\left.\left.+(-z_2-z_3+z_4)\,z_1+z_2^{\,2}-z_2z_4+z_3^{\,2}-z_3z_4+\frac{1}{2}\,z_4^{\,2}\right)\right)y_3^{\,2}-4\left(\left(\frac{1}{2}\,z_3-\frac{1}{2}\,z_2\right)y_4^{\,2}+(z_4\right.$$
$$\left.-z_1)\left(\frac{1}{2}\,z_1^{\,2}-z_2z_1+z_2^{\,2}+\left(-\frac{1}{2}\,z_3-\frac{1}{2}\,z_4\right)z_2+\frac{1}{2}\,z_3z_4\right)\right)(z_3-z_1)\,y_4y_3+2\,(z_3$$
$$-z_1)^{\,2}\left(\frac{1}{2}\,y_4^{\,4}+\left(\frac{1}{2}\,z_1^{\,2}+(-z_2+z_3-z_4)\,z_1+z_2^{\,2}-z_2z_3+\frac{1}{2}\,z_3^{\,2}-z_3z_4+z_4^{\,2}\right)y_4^{\,2}+\frac{1}{2}\,(z_4\right.$$
$$\left.-z_1)^{\,2}\,(z_3-z_4)^{\,2}\right)\right)x_2^{\,2}+\left(-2\,x_4\left(y_1^{\,2}+(-y_2-y_4)\,y_1+y_2y_4+(z_4-z_1)\,(z_2-z_1)\right)x_3^{\,4}\right.$$
$$+\left(\left(2\,y_1^{\,2}+(-2\,y_2-2\,y_4)\,y_1+2\,y_2y_4+2\,(z_4-z_1)\,(z_2-z_1)\right)x_4^{\,2}-2\left(y_1^{\,2}-2\,y_1y_4+y_4^{\,2}\right.\right.$$
$$\left.\left.+(z_4-z_1)^{\,2}\right)\left((-y_2+y_4)\,y_1+y_2^{\,2}-y_2y_4+(z_2-z_1)\,(z_2-z_4)\right)\right)x_3^{\,3}+2\left(\left(y_1^{\,2}+(-y_2\right.\right.$$
$$\left.\left.-y_3)\,y_1+y_2y_3+(z_3-z_1)\,(z_2-z_1)\right)x_4^{\,2}+(-y_2+2\,y_3-y_4)\,y_1^{\,3}+\left(y_2^{\,2}+(-y_3+2\,y_4)\,y_2\right.\right.$$
$$\left.\left.-2\,y_3^{\,2}-y_3y_4+y_4^{\,2}+(-z_2+2\,z_3-z_4)\,z_1+z_2^{\,2}+(z_3-2\,z_4)\,z_2-2\left(z_3+\frac{1}{2}\,z_4\right)(z_3-z_4)\right)y_1^{\,2}\right.$$
$$+\left((-y_3-y_4)\,y_2^{\,2}+\left(2\,y_3^{\,2}-y_4^{\,2}-z_1^{\,2}+(-2\,z_3+4\,z_4)\,z_1+2\,z_3^{\,2}-2\,z_3z_4-z_4^{\,2}\right)y_2+2\,y_3^{\,2}y_4\right.$$
$$\left.+\left(-y_4^{\,2}+2\,z_1^{\,2}+(-2\,z_2-2\,z_4)\,z_1-z_2^{\,2}+4\,z_2z_4-z_4^{\,2}\right)y_3-\left(z_1^{\,2}+(-4\,z_2+2\,z_3)\,z_1+z_2^{\,2}\right.\right.$$
$$\left.\left.+2\,z_2z_3-2\,z_3^{\,2}\right)y_4\right)y_1+\left(y_3y_4+(z_4-z_1)\,(z_3-z_1)\right)y_2^{\,2}+\left(-2\,y_3^{\,2}y_4+\left(y_4^{\,2}+(z_4-z_1)^{\,2}\right)y_3\right.$$
$$\left.-2\,y_4\,(z_3-z_1)^{\,2}\right)y_2+\left((2\,z_1-2\,z_4)\,y_3^{\,2}+y_4\,(z_2-z_1)\,y_3+(z_3-z_1)\left(y_4^{\,2}+(z_4-z_1)\,(z_2\right.\right.$$
$$\left.\left.-2\,z_3+z_4)\right)\right)(z_2-z_1)\Big)x_4x_3^{\,2}+\left(\left(-2\,y_1^{\,2}+(2\,y_2+2\,y_3)\,y_1-2\,y_2y_3-2\,(z_3-z_1)\,(z_2\right.\right.$$
$$\left.\left.-z_1)\right)x_4^{\,4}+\left((-2\,y_2-2\,y_3+4\,y_4)\,y_1^{\,3}+\left(2\,y_2^{\,2}+(4\,y_3-2\,y_4)\,y_2+2\,y_3^{\,2}-2\,y_3y_4-4\,y_4^{\,2}+(\right.\right.\right.$$
$$\left.\left.\left.-2\,z_2-2\,z_3+4\,z_4)\,z_1+2\,z_2^{\,2}+(-4\,z_3+2\,z_4)\,z_2+2\,(z_3+2\,z_4)\,(z_3-z_4)\right)y_1^{\,2}+\left((-2\,y_3\right.\right.\right.$$
$$\left.\left.\left.-2\,y_4)\,y_2^{\,2}+\left(-2\,y_3^{\,2}+4\,y_4^{\,2}-2\,z_1^{\,2}+(8\,z_3-4\,z_4)\,z_1-2\,z_3^{\,2}-4\,z_3z_4+4\,z_4^{\,2}\right)y_2-2\,y_3^{\,2}y_4\right.\right.$$
$$\left.\left.+\left(4\,y_4^{\,2}-2\,z_1^{\,2}+(8\,z_2-4\,z_4)\,z_1-2\,z_2^{\,2}-4\,z_2z_4+4\,z_4^{\,2}\right)y_3-2\left(-2\,z_1^{\,2}+(2\,z_2+2\,z_3)\,z_1+z_2^{\,2}\right.\right.\right.$$
$$\left.\left.\left.-4\,z_2z_3+z_3^{\,2}\right)y_4\right)y_1+\left(2\,y_3y_4+2\,(z_4-z_1)\,(z_3-z_1)\right)y_2^{\,2}+\left(2\,y_3^{\,2}y_4+\left(-4\,y_4^{\,2}-4\,(z_4\right.\right.\right.$$
$$\left.\left.\left.-z_1)^{\,2}\right)y_3+2\,y_4\,(z_3-z_1)^{\,2}\right)y_2+2\left((z_4-z_1)\,y_3^{\,2}+y_4\,(z_2-z_1)\,y_3+\left(-2\,y_4^{\,2}+(z_4-z_1)\,(z_2\right.\right.\right.$$
$$\left.\left.\left.+z_3-2\,z_4)\right)(z_3-z_1)\right)(z_2-z_1)\Big)x_4^{\,2}-2\left(y_1^{\,2}-2\,y_1y_4+y_4^{\,2}+(z_4-z_1)^{\,2}\right)\left(\left((y_3-y_4)\,y_2\right.\right.$$

$$-y_3y_4+y_4^2+(z_3-z_4)(z_2-z_4))y_1^2+((-y_3+y_4)y_2^2+(-y_3^2+2y_3y_4-y_4^2-(z_3$$
$$-z_4)^2)y_2+y_3^2y_4+(-y_4^2-(z_2-z_4)^2)y_3+y_4(-z_3+z_2)^2)y_1+(y_3^2-y_3y_4+(z_3-z_1)(z_3$$
$$-z_4))y_2^2+(-y_3^2y_4+(y_4^2+(z_4-z_1)^2)y_3-y_4(z_3-z_1)^2)y_2+((z_2-z_4)y_3^2-y_4(z_2$$
$$-z_1)y_3+(z_3-z_1)(y_4^2+(z_3-z_4)(z_2-z_4)))(z_2-z_1)))x_3+2x_4(y_1^2-2y_1y_3+y_3^2$$
$$+(z_3-z_1)^2)(((y_2-y_3)y_1-y_2^2+y_2y_3-(z_2-z_1)(-z_3+z_2))x_4^2+((y_3-y_4)y_2-y_3^2$$
$$+y_3y_4+(z_3-z_4)(-z_3+z_2))y_1^2+((-y_3+y_4)y_2^2+(y_3^2-2y_3y_4+y_4^2+(z_3-z_4)^2)y_2$$
$$+y_3^2y_4+(-y_4^2-(z_2-z_4)^2)y_3+y_4(-z_3+z_2)^2)y_1+(y_3y_4-y_4^2+(z_4-z_1)(z_3-z_4))y_2^2$$
$$+(-y_3^2y_4+(y_4^2+(z_4-z_1)^2)y_3-y_4(z_3-z_1)^2)y_2+((-z_4+z_1)y_3^2+y_4(z_2-z_1)y_3+($$
$$-y_4^2+(z_4-z_1)(z_3-z_4))(-z_3+z_2))(z_2-z_1)))x_2+((y_1^2-2y_1y_2+y_2^2+(z_2-z_1)^2)x_4^2$$
$$+((z_2-z_4)y_1+(z_4-z_1)y_2-y_4(z_2-z_1))^2)x_3^4+2(-x_4^2+(-y_2+y_4)y_1+y_2y_4-y_4^2$$
$$+(z_4-z_1)(z_2-z_4))(y_1^2-2y_1y_2+y_2^2+(z_2-z_1)^2)x_4x_3^3+((y_1^2-2y_1y_2+y_2^2$$
$$+(z_2-z_1)^2)x_4^4+((4y_2-2y_3-2y_4)y_1^3+(-8y_2^2+(2y_3+2y_4)y_2+2y_3^2+2y_4^2+(4z_2$$
$$-2z_3-2z_4)z_1-2z_2^2+2z_3^2-2z_3z_4+2z_4^2)y_1^2+(4y_2^3+(2y_3+2y_4)y_2^2+(-4y_3^2$$
$$-4y_4^2+4z_1^2+(-12z_2+2z_3+2z_4)z_1+4z_2^2+(2z_3+2z_4)z_2-4z_3^2+4z_3z_4-4z_4^2)y_2$$
$$-2((z_4-z_1)y_3+y_4(z_3-z_1))(z_2-z_1))y_1+(-2y_3-2y_4)y_2^3+(2y_3^2+2y_4^2-2z_1^2$$
$$+4z_2z_1+(-2z_3-2z_4)z_2+2z_3^2-2z_3z_4+2z_4^2)y_2^2-2(z_2-z_1)((z_2-z_4)y_3+y_4(-z_3$$
$$+z_2))y_2-2(-y_3^2+y_3y_4-y_4^2+(-2z_2+z_3+z_4)z_1+(z_3+z_4)z_2-z_3^2-z_4^2)(z_2-z_1)^2)$$
$$x_4^2+(y_2^2-2y_2y_4+y_4^2+(z_2-z_4)^2)y_1^4+(-2y_2^3+2y_2^2y_4+(2y_4^2-2(z_2-z_4)(-z_3$$
$$+z_2))y_2-2(z_2-z_4)^2y_3-2y_4(y_4^2+(z_3-z_4)(z_2-z_4)))y_1^3+(y_2^4+2y_2^3y_4+(-6y_4^2$$
$$+2z_1^2+(-2z_2-2z_3)z_1+2z_2^2-2z_2z_3+4z_3z_4-2z_4^2)y_2^2+(-2(z_4-z_1)(z_2-z_4)y_3$$
$$+2y_4(y_4^2-2z_1^2+(z_2+2z_3+z_4)z_1+z_2^2+(-z_3-2z_4)z_2-z_4(z_3-z_4)))y_2+2(z_2$$
$$-z_4)^2y_3^2+2y_4(z_2-z_1)(z_2-z_4)y_3+y_4^4+(2z_1^2+(-2z_3-2z_4)z_1-2z_2^2+4z_2z_3$$

$$-2z_4(z_3-z_4))y_4^2+(z_2-z_4)^2(2z_1^2+(-2z_2-2z_4)z_1+z_2^2+(-2z_3+2z_4)z_2+2z_3^2$$
$$-2z_3z_4+z_4^2))y_1^2+\Big(-2y_2^4y_4+\big(2y_4^2-2(z_4-z_1)(z_3-z_1)\big)y_2^3+\Big(-2(z_4-z_1)(z_2$$
$$-z_4)y_3-4\Big(-\frac{1}{2}y_4^2-\frac{1}{2}z_1^2+\Big(-\frac{1}{2}z_2+\frac{1}{2}z_3+z_4\Big)z_1+z_2^2+\Big(-z_3-\frac{1}{2}z_4\Big)z_2+\frac{1}{2}z_4(z_3$$
$$-z_4)\Big)y_4\Big)y_2^2+\big(4(z_4-z_1)(z_2-z_4)y_3^2-2y_4^4+\big(2z_1^2+(-4z_2-2z_3+2z_4)z_1+2z_2^2$$
$$+(-2z_3+2z_4)z_2+4z_4(z_3-z_4)\big)y_4^2-2(z_4-z_1)(z_2-z_4)\big((-z_2+z_3)z_1+z_2z_3-2z_3^2$$
$$+2z_3z_4-z_4^2)\big)y_2-2\big(2y_4(z_2-z_4)y_3^2+(z_2-z_4)\big(-y_4^2+(z_4-z_1)(z_2-z_4)\big)y_3+\big((z_3$$
$$-z_1)y_4^2+(z_2-z_4)\big((-z_3+z_4)z_1+z_2^2-2z_2z_3+2z_3^2-z_3z_4\big)\big)y_4\big)(z_2-z_1)\big)y_1+\big(y_4^2$$
$$+(z_4-z_1)^2\big)y_2^4+\big(-2(z_4-z_1)^2y_3+2\big(-y_4^2+(z_4-z_1)(z_3-z_4)\big)y_4\big)y_2^3+\Big(2(z_4$$
$$-z_1)^2y_3^2+2y_4(z_4-z_1)(z_2-z_1)y_3+y_4^4+\big(-2z_1^2+4z_3z_1+2z_2^2+(-2z_3-2z_4)z_2$$
$$-2z_4(z_3-z_4)\big)y_4^2+2(z_4-z_1)^2\Big(\frac{1}{2}z_1^2+(-z_2-z_3+z_4)z_1+z_2^2-z_2z_4+z_3^2-z_3z_4$$
$$+\frac{1}{2}z_4^2\Big)\Big)y_2^2+2\big(-2y_4(z_4-z_1)y_3^2-(z_4-z_1)\big(-y_4^2+(z_4-z_1)(z_2-z_4)\big)y_3+\big((-z_2$$
$$+z_3)y_4^2+(z_4-z_1)\big(-z_1^2+2z_3z_1+(z_3-z_4)z_2-2z_3^2+z_3z_4\big)\big)y_4\big)(z_2-z_1)y_2+\big(2y_3^2y_4^2$$
$$+2\big(-y_4^2+(z_4-z_1)(z_2-z_4)\big)y_4y_3+y_4^4+\big(z_1^2+(2z_2-2z_3-2z_4)z_1+z_2^2+(-2z_3$$
$$-2z_4)z_2+2z_3^2+2z_4^2\big)y_4^2+(z_4-z_1)^2(z_2-z_4)^2\big)(z_2-z_1)^2\big)x_3^2-2\big(y_1^2-2y_1y_2+y_2^2$$
$$+(z_2-z_1)^2\big)x_4\Big(\big((y_2-y_3)y_1-y_2y_3+y_3^2-(z_3-z_1)(-z_3+z_2)\big)x_4^2+\big(y_2^2+(-y_3$$
$$-y_4)y_2+y_3y_4+(z_2-z_4)(-z_3+z_2)\big)y_1^2+\big((-y_3-y_4)y_2^2+\big(y_3^2+2y_3y_4+y_4^2+(z_3$$
$$-z_4)^2\big)y_2-y_3^2y_4+\big(-y_4^2-(z_2-z_4)^2\big)y_3-y_4(-z_3+z_2)^2\big)y_1+\big(y_3y_4+(z_4-z_1)(z_3$$
$$-z_1)\big)y_2^2+\big(-y_3^2y_4+\big(-y_4^2-(z_4-z_1)^2\big)y_3-y_4(z_3-z_1)^2\big)y_2+\big(y_4^2-(z_4-z_1)(z_2$$
$$-z_4)\big)y_3^2+y_4(z_2-z_1)^2y_3+(-z_3+z_2)(z_3-z_1)\big(-y_4^2+(z_4-z_1)(z_2-z_4)\big)\big)x_3+\big(($$
$$-z_3+z_2)y_1+(z_3-z_1)y_2-y_3(z_2-z_1)\big)^2x_4^4+\Big(\big(y_2^2-2y_2y_3+y_3^2+(-z_3+z_2)^2\big)y_1^4+\big($$
$$-2y_2^3+2y_2^2y_3+\big(2y_3^2-2(z_2-z_4)(-z_3+z_2)\big)y_2-2y_3^3+2(z_3-z_4)(-z_3+z_2)y_3$$
$$-2y_4(-z_3+z_2)^2\big)y_1^3+\big(y_2^4+2y_2^3y_3+\big(-6y_3^2+2z_1^2+(-2z_2-2z_4)z_1+2z_2^2-2z_2z_4$$

$$-2z_3(z_3-2z_4))y_2^2+(2y_3^3+(-4z_1^2+(2z_2+2z_3+4z_4)z_1+2z_2^2+(-4z_3-2z_4)z_2$$
$$+2z_3(z_3-z_4))y_3-2y_4(z_3-z_1)(-z_3+z_2))y_2+y_3^4+(2z_1^2+(-2z_3-2z_4)z_1-2z_2^2$$
$$+4z_2z_4+2z_3(z_3-z_4))y_3^2+2y_4(z_2-z_1)(-z_3+z_2)y_3+(-z_3+z_2)^2(2y_4^2+2z_1^2+(-2z_2$$
$$-2z_3)z_1+z_2^2+(2z_3-2z_4)z_2+z_3^2-2z_3z_4+2z_4^2))y_1^2+(-2y_2^4y_3+(2y_3^2-2(z_4$$
$$-z_1)(z_3-z_1))y_2^3+(2y_3^3+(2z_1^2+(2z_2-4z_3-2z_4)z_1-4z_2^2+(2z_3+4z_4)z_2$$
$$+2z_3(z_3-z_4))y_3-2y_4(z_3-z_1)(-z_3+z_2))y_2^2+(-2y_3^4+(2z_1^2+(-4z_2+2z_3$$
$$-2z_4)z_1+2z_2^2+(2z_3-2z_4)z_2-4z_3(z_3-z_4))y_3^2-2(-2y_4^2+(-z_2+z_4)z_1+z_2z_4-z_3^2$$
$$+2z_3z_4-2z_4^2)(-z_3+z_2)(z_3-z_1))y_2-2((z_4-z_1)y_3^3-y_4(-z_3+z_2)y_3^2+(2y_4^2+(z_3$$
$$-z_4)z_1+z_2^2-2z_2z_4-z_4(z_3-2z_4))(-z_3+z_2)y_3+y_4(-z_3+z_2)^2(z_3-z_1))(z_2-z_1))y_1$$
$$+(y_3^2+(z_3-z_1)^2)y_2^4+(-2y_3^3-2(z_3-z_1)(z_3-z_4)y_3-2y_4(z_3-z_1)^2)y_2^3+(y_3^4+($$
$$-2z_1^2+4z_4z_1+2z_2^2+(-2z_3-2z_4)z_2+2z_3(z_3-z_4))y_3^2+2y_4(z_3-z_1)(z_2-z_1)y_3$$
$$+2(z_3-z_1)^2(y_4^2+\frac{1}{2}z_1^2+(-z_2+z_3-z_4)z_1+z_2^2-z_2z_3+\frac{1}{2}z_3^2-z_3z_4+z_4^2))y_2^2$$
$$-2((z_2-z_4)y_3^3-y_4(z_3-z_1)y_3^2+(z_3-z_1)(2y_4^2+z_1^2-2z_4z_1+(z_3-z_4)z_2-z_4(z_3$$
$$-2z_4))y_3+y_4(z_3-z_1)^2(-z_3+z_2))(z_2-z_1)y_2+(y_3^4-2y_3^3y_4+(2y_4^2+z_1^2+(2z_2$$
$$-2z_3-2z_4)z_1+z_2^2+(-2z_3-2z_4)z_2+2z_3^2+2z_4^2)y_3^2+2y_4(z_3-z_1)(-z_3+z_2)y_3$$
$$+(z_3-z_1)^2(-z_3+z_2)^2)(z_2-z_1)^2)x_4^2+(((-z_3+z_4)y_2+(z_2-z_4)y_3-y_4(-z_3$$
$$+z_2))y_1^2+((z_3-z_4)y_2^2+(-z_2+z_4)y_3^2+(-z_3+z_2)(y_4^2+(z_3-z_4)(z_2-z_4)))y_1$$
$$+((z_4-z_1)y_3-y_4(z_3-z_1))y_2^2+((-z_4+z_1)y_3^2-(-y_4^2+(z_4-z_1)(z_3-z_4))(z_3$$
$$-z_1))y_2-(-y_3^2y_4+(y_4^2-(z_4-z_1)(z_2-z_4))y_3+y_4(z_3-z_1)(-z_3+z_2))(z_2-z_1))^2)^{1/2}$$
$$/(((-2z_3+2z_4)y_2+(2z_2-2z_4)y_3-2y_4(-z_3+z_2))x_1+((2z_3-2z_4)y_1+(2z_4$$
$$-2z_1)y_3-2y_4(z_3-z_1))x_2+((-2z_2+2z_4)y_1+(2z_1-2z_4)y_2+2y_4(z_2-z_1))x_3$$
$$+2x_4((-z_3+z_2)y_1+(z_3-z_1)y_2-y_3(z_2-z_1)))$$

Define Four Random Points and Calculate Coordinates and Radius of Sphere

Generate four random points

```
> pars := Equate([x1, y1, z1, x2, y2, z2, x3, y3, z3, x4, y4, z4], RandomTools:-Generate(list(float, 12)))
```

$$pars := [x_1 = 0.1385826213, y_1 = 0.6070209412, z_1 = 0.4192704615, x_2 = 0.6219707632, y_2 = 0.4191647709, z_2 = 0.7381384754, x_3 = 0.8522737115, y_3 = 0.9194600998, z_3 = 0.9926182002, x_4 = 0.2451507579, y_4 = 0.6257194611, z_4 = 0.7667276150] \quad (4.1)$$

Hence the coordinates of the center and radius of a sphere whose surface passes through all four points and radius

```
> coords := eval([res1, res2, res3, R], pars)
```

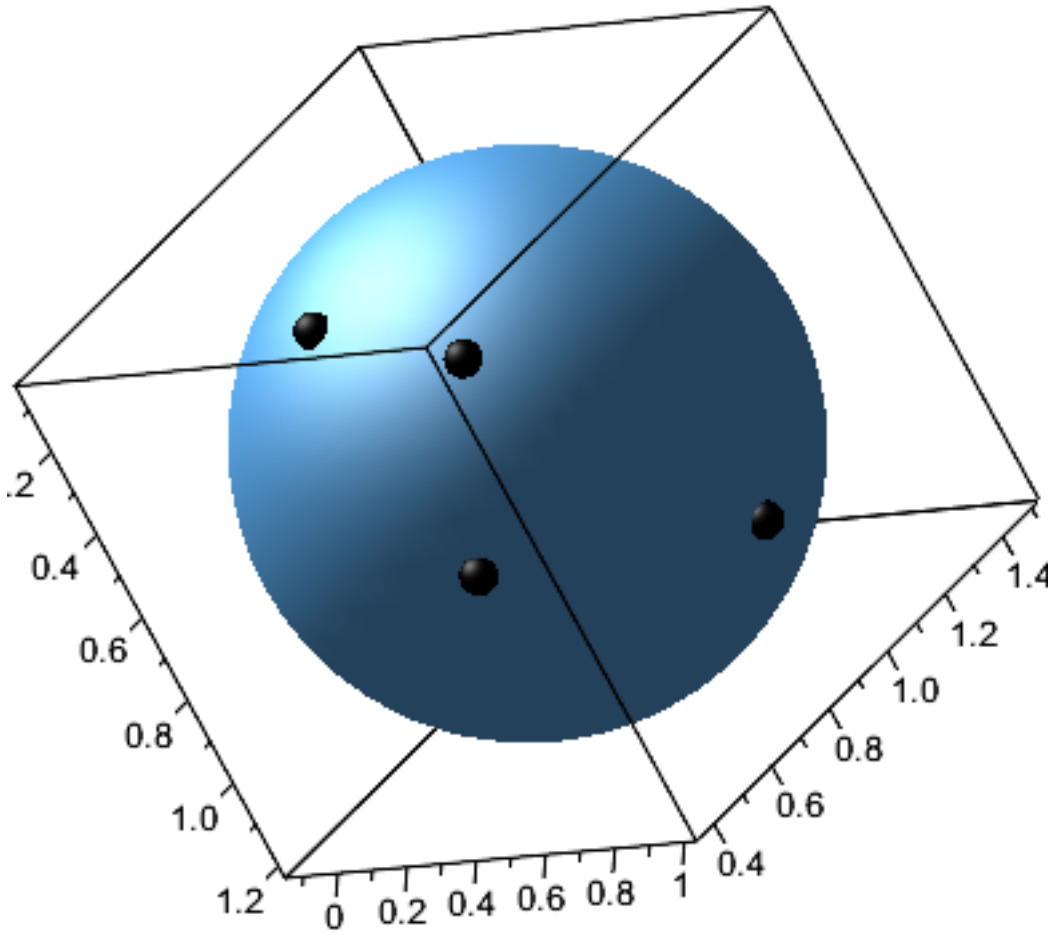
$$coords := [x_c = 0.6347692725, y_c = 0.9327345125, z_c = 0.4401317074, r = 0.5939071918] \quad (4.2)$$

Visualize the four random points and sphere

```
> p1 := plottools:-sphere(rhs~(coords[1..-2]), rhs~(coords[-1]), style=surface, color
      ="SteelBlue") :

  p2 := plottools:-sphere(rhs~(pars[1 .. 3]), rhs~(coords[-1])/15, style=surface, color=black)
    , plottools:-sphere(rhs~(pars[4 .. 6]), rhs~(coords[-1])/15, style=surface, color=black)
    , plottools:-sphere(rhs~(pars[7 .. 9]), rhs~(coords[-1])/15, style=surface, color=black)
    , plottools:-sphere(rhs~(pars[10 .. 12]), rhs~(coords[-1])/15, style=surface, color=black) :

  plots:-display(p1, p2, scaling=constrained, axesfont=[Arial, 10])
```




```
>
```